Math 132: Differential Topology

§ Stokes theorem

Stokes theorem is a remarkable theorem relating d with ∂, which generalizes the fundamental theorem of calculus.

Thm (Stokes) Let M be a compact, oriented m-manifold with boundary. Then, for any smooth $(m-1)$-form ω on M,

$$\int_{\partial M} \omega = \int_M d\omega .$$

Before getting into the proof, let's first appreciate this theorem for a moment.

Rmk
- When $M = [a,b] \subset \mathbb{R}$ and $\omega : [a,b] \to \mathbb{R}$ is a function, this is just the fundamental theorem of calculus.
- When M is boundaryless, we see that the integral of any exact form on a boundaryless manifold is 0.
- If ω is a closed $(m-1)$-form on M, then $\int_{\partial M} \omega = 0$.
- Note also that the theorem is compatible with $\partial^2 = 0$ and $d^2 = 0$:

$$0 = \int_{\partial^2 M} \omega = \int_{\partial M} d\omega = \int_M d^2\omega = 0 .$$

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proof of the thm)

Since both sides of the equation are linear in ω, we may assume that ω has a compact support, contained in a parametrizable open subset U. That is, we can work locally in an open subset of $\mathbb{R}^m$ or H^m.

Any $(m-1)$-form in m-space may be written as

$$\nu = \sum_{i=1}^{m} (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m$$

$\widehat{dx_i}$ means dx_i is omitted

So that $d\nu = \left(\sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_m$.

If U is open in $\mathbb{R}^m$ and ν has compact support in U, then

$$\int_U d\nu = \sum_i \int_{\mathbb{R}^m} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_m \underset{\text{Fubini}}{=} \sum_i \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_m$$

$$\underset{\text{fundamental thm of calculus}}{=} 0 \quad , \quad \text{as desired.}$$

If U is open in H^m, then

$$\int_U d\nu = \sum_i \int_{H^m} \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_m \overset{\text{Fubini}}{=} \int_{\mathbb{R}^{m-1}} \left(\int_0^{\infty} \frac{\partial f_m}{\partial x_m} dx_m \right) dx_1 \cdots dx_{m-1}$$

$$= \int_{\mathbb{R}^{m-1}} -f_m(x_1, \cdots, x_{m-1}, 0)\, dx_1 \cdots dx_{m-1} =: \star$$

fundamental thm of calculus.

On the other hand, since $x_m = 0$ on ∂H^m, $dx_m = 0$ on ∂H^m, and it follows that

$$\int_{\partial H^m} \nu = \int_{\partial H^m} (-1)^{m-1} f_m(x_1, \cdots, x_{m-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{m-1}$$

$$= (-1)^m \int_{\mathbb{R}^{m-1}} (-1)^{m-1} f_m(x_1, \cdots, x_{m-1}, 0)\, dx_1 \cdots dx_{m-1} = \star$$

as desired. ■

change of orientation under diffeo $\partial H^m \longrightarrow \mathbb{R}^{m-1}$, $(x_1, \cdots, x_{m-1}, 0) \mapsto (x_1, \cdots, x_{m-1})$

which is given by the sign of the ordered basis $\{-e_m, e_1, \cdots, e_{m-1}\}$.

Cor If P is a p-dim'l compact, oriented submanifold of M, then integral over P defines a linear functional

$$\int_P : H^p(M) \longrightarrow \mathbb{R}.$$

Moreover, if $P = \partial W$ for some compact, oriented $(p+1)$-dim'l submanifold with ∂ in M, then $\int_P = 0$.

<u>Cor</u> If $f: M \to N$ is a smooth map of compact, oriented m-mflds which extends to all of a compact, oriented $(m+1)$-mfld W with bdry $\partial W = M$, then $\int_M f^* \omega = 0$ for any m-form ω on N.

pf) $\displaystyle\int_M f^*\omega = \int_{\partial W} F^*\omega \overset{\text{Stokes}}{=} \int_W F^* d\omega = 0$

$F: W \to N$; since ω is a top degree form.

<u>Cor</u> If $f_0, f_1 : M \to N$ are homotopic maps of compact oriented m-mflds, then for every m-form ω on N, $\int_M f_0^*\omega = \int_M f_1^*\omega$.

pf) Apply the previous corollary to the homotopy $F: I \times M \to N$.

<u>Rmk</u> While we won't have time to cover this, in fact the following <u>degree formula</u> holds:

$$\int_M f^*\omega = \deg(f) \int_N \omega .$$